

# SOME REMARKS CONCERNING SYMMETRY-BREAKING FOR THE GINZBURG-LANDAU EQUATION

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**ABSTRACT.** The correlation term, introduced in [13] to describe the interaction between very far apart vortices, governs symmetry-breaking for the Ginzburg-Landau equation in  $\mathbb{R}^2$  or bounded domains. It is a homogeneous function of degree  $(-2)$ , and then for  $\frac{2\pi}{N}$ -symmetric vortex configurations can be expressed in terms of the so-called correlation coefficient. Ovchinnikov and Sigal [13] have computed it in few cases and conjectured its value to be an integer multiple of  $\frac{\pi}{4}$ . We will disprove this conjecture by showing that the correlation coefficient always vanishes, and will discuss some of its consequences.

*Keywords:* Ginzburg-Landau equation, Symmetry-breaking, correlation term

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## 1. INTRODUCTION

The Ginzburg-Landau theory is a very popular model in super-conductivity [6]. Stationary states are described by complex-valued solutions  $u$  of the planar equation

$$-\Delta u = k^2 u(1 - |u|^2),$$

where  $k > 0$  is the Ginzburg-Landau parameter. The condensate wave function  $u$  describes the superconductive regime in the sample by simply interpreting  $|u|^2$  as the density of Cooper electrons pairs. The zeroes of  $u$ , where the normal state is restored, are called vortices. The parameter  $k$  depends on the physical properties of the material and distinguishes between Type I superconductors  $k < \frac{1}{\sqrt{2}}$  (in this normalization of constants) and Type II superconductors  $k > \frac{1}{\sqrt{2}}$ .

In the entire plane  $\mathbb{R}^2$  the parameter  $k$  does not play any role, as we can reduce to the case  $k = 1$  by simply changing  $u$  into  $u(\frac{x}{k})$ . Supplemented by the correct asymptotic behavior at infinity, the Ginzburg-Landau equation now reads as

$$\begin{cases} -\Delta U = U(1 - |U|^2) & \text{in } \mathbb{R}^2 \\ |U| \rightarrow 1 \text{ as } |x| \rightarrow \infty. \end{cases} \quad (1.1)$$

The condition  $|U| \rightarrow 1$  as  $|x| \rightarrow \infty$  allows to define the (topological) degree  $\deg U$  of  $U$  as the winding number of  $U$  at  $\infty$ :

$$\deg U = \frac{1}{2\pi} \int_{|x|=R} d(\arg U),$$

where  $R > 0$  is chosen large so that  $|U| \geq \frac{1}{2}$  in  $\mathbb{R}^2 \setminus B_R(0)$ . Given  $n \in \mathbb{Z}$ , the only known solution of (1.1) with  $\deg U = n$  is the “radially symmetric” one  $U_n(x) = S_n(|x|)(\frac{x}{|x|})^n$  (in complex notations with  $x \in \mathbb{C}$ ), where  $S_n$  is the solution of the following ODE:

$$\begin{cases} \ddot{S}_n + \frac{1}{r}\dot{S}_n - \frac{n^2}{r^2}S_n + S_n(1 - S_n^2) = 0 & \text{in } (0, +\infty) \\ S_n(0) = 0, \quad \lim_{r \rightarrow +\infty} S_n = 1. \end{cases}$$

Existence and uniqueness of  $S_n$  is shown in [4, 7]. Moreover, the solution  $U_n$  is stable for  $|n| \leq 1$  and unstable for  $|n| > 1$  [11]. When  $n = \pm 1$ , the solution  $U_{\pm 1}$  is unique, modulo translations and rotations, in the class of functions  $U$  with  $\deg U = \pm 1$  and  $\int_{\mathbb{R}^2} (|U|^2 - 1)^2 dx < +\infty$  [10].

One of the open problems (Problem 1)– that Brezis-Merle-Rivi re raise out in [3]– concerns the existence of solutions  $U$  of (1.1) with  $\deg U = n$ ,  $|n| > 1$ , which are not “radially symmetric” around any point. So far there is no rigorous answer, but a strategy to find them has been proposed in [12]. Formally, a solution  $U$  of (1.1) is a critical point of the functional

$$\mathcal{E}(\Psi) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla \Psi|^2 dx + \frac{1}{4} \int_{\mathbb{R}^2} (|\Psi|^2 - 1)^2 dx.$$

Since  $\mathcal{E}(\Psi) = +\infty$  for any  $C^1$ -map  $\Psi$  so that  $|\Psi| \rightarrow 1$  as  $|x| \rightarrow +\infty$  and  $\deg(\Psi) \neq 0$ , Ovchinnikov and Sigal [11] have proposed to correct  $\mathcal{E}$  into

$$\mathcal{E}_{\text{ren}}(\Psi) = \int_{\mathbb{R}^2} \left( \frac{1}{2} |\nabla \Psi|^2 - \frac{(\deg \Psi)^2}{|x|^2} \chi + \frac{1}{4} (|\Psi|^2 - 1)^2 \right) dx,$$

where  $\chi$  is a smooth cut-off function with  $\chi = 0$  when  $|x| \leq R$  and  $\chi = 1$  when  $|x| \geq R + R^{-1}$ , and  $R \gg 1$  is given. Given a vortex configuration  $(\underline{a}, \underline{n}) = (a_1, \dots, a_K, n_1, \dots, n_K)$ , a  $C^1$ -map  $\Psi$  so that  $|\Psi| \rightarrow 1$  as  $|x| \rightarrow +\infty$  has vortex configuration  $(\underline{a}, \underline{n})$  if  $a_1, \dots, a_K$  are the only zeroes of  $\Psi$  with local indices  $n_1, \dots, n_K$ , denoted for short as  $\text{conf } \Psi = (\underline{a}, \underline{n})$ . Given  $\underline{n}_0$ , Ovchinnikov and Sigal [12] introduce the “intervortex energy”  $E$  given by

$$E(\underline{a}) = \inf \{ \mathcal{E}_{\text{ren}}(\Psi) : \text{conf } \Psi = (\underline{a}, \underline{n}_0) \},$$

and conjecture that  $\underline{a}_0$  is a critical point of  $E$  if and only if there is a minimizer  $U$  for  $E(\underline{a}_0)$ , yielding to a solution of (1.1) with  $\text{conf } U = (\underline{a}_0, \underline{n}_0)$  which is not “radially symmetric” around any point by construction. Letting  $d_{\underline{a}} = \min_{i \neq j} |a_i - a_j|$ , the following asymptotic expression is established [12]:

$$E(\underline{a}) = \sum_{j=1}^K \mathcal{E}_{\text{ren}}(U_{n_j}) + H\left(\frac{\underline{a}}{R}\right) + \text{Rem} \quad (1.2)$$

with  $\text{Rem} = O(d_{\underline{a}}^{-1})$  as  $d_{\underline{a}} \rightarrow +\infty$ , where  $H(\underline{a}) = -\pi \sum_{i \neq j} n_i n_j \ln |a_i - a_j|$  is the energy of the vortex pairs interactions. When  $\nabla H(\underline{a}) = 0$ , the estimate in (1.2) improves up to  $\text{Rem} = O(d_{\underline{a}}^{-2})$ .

When  $\nabla H(\underline{a}) = 0$  (a so-called forceless vortex configuration), by choosing refined test functions the asymptotic expression (1.2) is improved [13] in the form of the following upper bound:

$$E(\underline{a}) \leq \sum_{j=1}^K \mathcal{E}_{\text{ren}}(U_{n_j}) + H\left(\frac{\underline{a}}{R}\right) - A(\underline{a}) + \text{Rem} \quad (1.3)$$

with  $\text{Rem} = O(d_{\underline{a}}^{-2} + R^{-2})$  as  $d_{\underline{a}} \rightarrow +\infty$ , where the correlation term  $A(\underline{a})$  is a homogeneous function of degree  $(-2)$  given as

$$A(\underline{a}) = \frac{1}{4} \int_{\mathbb{R}^2} \left[ \left| \sum_{j=1}^K \nabla \varphi_j \right|^4 - \sum_{j=1}^K |\nabla \varphi_j|^4 \right],$$

with  $\varphi_j(x) = n_j \theta(x - a_j)$ ,  $j = 1, \dots, K$ , and  $\theta(x)$  the polar angle of  $x \in \mathbb{R}^2$ .

To push further the analysis, in [13] the attention is restricted to symmetric vortex configurations in order to reduce the number of independent variables in  $E(\underline{a})$ . In particular, the simplest  $\frac{2\pi}{N}$ -symmetric vortex configurations  $(\underline{a}, \underline{n})$  (which are invariant under  $\frac{2\pi}{N}$ -rotations

and reflections w.r.t. the real axis) have the form:  $a_0 = 0$ ,  $a_1, \dots, a_N$  are the vertices of a regular  $N$ -polygon with  $a_1 = 1$  and  $n_1 = \dots = n_N = m$ . We impose also the forceless condition  $\nabla H(\underline{a}) = 0$ , which simply reads as  $n_0 = -\frac{N-1}{2}m$ .

Since  $|a_1| = \dots = |a_N|$ , the only variable is the size  $a = |a_1|$  of the polygon, and then the intervortex energy will be in the form  $E(a)$ . Since  $A(\underline{a})$  is homogeneous of degree  $-2$ , we have that  $A(\underline{a}) = \frac{A_0}{a^2}$ , where

$$A_0 := A(1, e^{\frac{2\pi i}{N}}, \dots, e^{\frac{2\pi i(N-1)}{N}}) \quad (1.4)$$

is the correlation coefficient for given  $n_0 = -\frac{N-1}{2}m$  and  $n_1 = \dots = n_N = m$ . In [13] the existence of c.p.'s of  $E(a)$  is shown for the cases  $(N, m) = (2, 2)$  and  $(N, m) = (4, 2)$  by comparing  $E(a)$  for  $a$  small and large, and using the positive sign of  $A_0$  (the correlation coefficient has value  $8\pi$  and  $80\pi$ , respectively). It is also conjectured [13] that the correlation coefficient has values which are integer multiples of  $\frac{\pi}{4}$ . With a long but tricky computation, in the next section we will disprove such a conjecture by showing

**Theorem 1.1.** *The correlation coefficient in (1.4) always vanishes:  $A_0 = 0$ , for all  $N \geq 2$  and  $m \in \mathbb{Z}$ .*

Beside the role of  $A_0$  in symmetry-breaking phenomena for (1.1) in  $\mathbb{R}^2$ , as already discussed, we will also explain its connection with the Ginzburg-Landau equation

$$\begin{cases} -\Delta u = k^2 u(1 - |u|^2) & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \quad (1.5)$$

on a bounded domain  $\Omega$  for strongly Type II superconductors  $k \rightarrow +\infty$ , where  $g : \partial\Omega \rightarrow S^1$  is a smooth map.

The energy functional for (1.5)

$$E_k(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 + \frac{k^2}{4} \int_{\Omega} (1 - |u|^2)^2$$

has always a minimizer  $\bar{u}_k$  in the space  $H = \{u \in H^1(\Omega, \mathbb{C}) : u = g \text{ on } \partial\Omega\}$ . When  $d = \deg g \neq 0$ , by [2, 15, 16] we know that on simply connected domains  $\bar{u}_k$  has exactly  $|d|$  simple zeroes  $a_1, \dots, a_{|d|}$  for  $k$  large, where  $(a_1, \dots, a_{|d|})$  is a critical point for a suitable “renormalized energy”  $W(a_1, \dots, a_{|d|})$ . The symmetry-breaking phenomenon here takes place, driven by an external mechanism like the boundary condition that forces the confinement of vortices in some equilibrium configuration. A similar result does hold [2] on star-shaped domains for any solutions sequence  $u_k$  of (1.5). Near any vortex  $a_i$ , the function  $u(\frac{x}{k} + a_i)$  behaves like  $U_{n_i}(x)$ .

Once the asymptotic behavior is well understood, a natural question concerns the construction of such solutions for any given c.p.  $(a_1, \dots, a_K)$  of  $W$ , and a positive answer has been given by a heat-flow approach [8, 9], by topological methods [1] and by perturbative methods [5, 14] in case  $n_1 = \dots = n_K = \pm 1$ . In [14], page 12, it is presented as an open problem to know whether or not there are solutions having vortices collapsing as  $k \rightarrow \infty$ , the simplest situation being problem (1.5) on the unit ball  $B$  with boundary value  $g_0 = \frac{x^2}{|x|^2}$ :

$$\begin{cases} -\Delta u = k^2 u(1 - |u|^2) & \text{in } B \\ u = g_0 & \text{on } \partial B. \end{cases} \quad (1.6)$$

It is conjectured the existence of solutions to (1.6) having a vortex of degree  $-1$  at the origin  $a_0 = 0$  and three vortices of degrees  $+1$  at the vertices  $la_j$ ,  $a_j = e^{\frac{2\pi i}{3}(j-1)}$  for  $j = 1, 2, 3$ , of a

small ( $l \ll 1$ ) equilateral triangle centered at 0. This vortex configuration is  $\frac{2\pi}{3}$ -symmetric, forceless and has “renormalized energy”

$$W(l) = -6\pi \ln 3 - 6\pi \ln(1 - l^6) + O(l^9), \quad l > 0. \quad (1.7)$$

In collaboration with J. Wei, we were working on this problem. Inspired by [5], we were aiming to use a reduction argument of Lyapunov-Schmidt type, starting from the approximating solutions  $U_k$  for (1.6) given by

$$U_k(x) = e^{i\varphi_k(x)} U_{-1}(kx) \prod_{j=1}^3 U_1 \left( k(x - l e^{\frac{2\pi i}{3}(j-1)}) \right)$$

with  $l \rightarrow 0$  and  $lk \rightarrow +\infty$ , where the function  $\varphi_k$  is an harmonic function so that  $U_k|_{\partial B} = g_0$ . The interaction due to the collapsing of three vortices onto 0 gives at main order a term  $(lk)^{-2}$  with the plus sign, i.e. for some  $J_0 > 0$  there holds the energy expansion

$$\begin{aligned} E_k(U_k) &= 4\pi \ln k + I + \frac{1}{2}W(l) + J_0(lk)^{-2} + o((lk)^{-2}) \\ &= 4\pi \ln k + I - 3\pi \ln 3 + 3\pi l^6 + J_0(lk)^{-2} + o(l^6 + (lk)^{-2}), \end{aligned} \quad (1.8)$$

in view of (1.7). The aim is to construct a solution  $u_k$  in the form  $U_k[\eta(1 + \psi) + (1 - \eta)e^\psi]$ , where  $\psi = \psi(k)$  is a remainder term small in a weighted  $L^\infty(B)$ -norm and  $l = l(k)$  as  $k \rightarrow +\infty$ . The function  $\eta$  is a smooth cut-off function with  $\eta = 1$  in  $\cup_{j=0}^3 B_{1/k}(la_j)$  and  $\eta = 0$  in  $B \setminus \cup_{j=0}^3 B_{2/k}(la_j)$ . The function  $\psi = \psi(k)$  is found thanks to the solvability theory (up to a finite-dimensional kernel) of the linearized operator for (1.6) at  $U_k$  as  $l \rightarrow 0$  and  $lk \rightarrow +\infty$ , and by the Lyapunov-Schmidt reduction the existence of  $l(k)$  follows as a c.p. of

$$\tilde{E}_k := E_k(U_k[\eta(1 + \psi(k)) + (1 - \eta)e^{\psi(k)}]).$$

If  $U_k$  is sufficiently good as an approximating solution of (1.6), we have that  $\tilde{E}_k = E_k(U_k) + o((lk)^{-2})$ . Since  $3\pi l^6 + J_0(lk)^{-2}$  has always a minimum point of order  $k^{-\frac{1}{4}}$  as  $k \rightarrow +\infty$ , by (1.8) we get the existence of  $l = l(k)$  in view of the persistence of minimum points under small perturbations.

Unfortunately, this is not the case. Pushing further the analysis, we were able to identify the leading term  $\psi_0 = \psi_0(k)$  of  $\psi = \psi(k)$ , and compute its contribution into the energy expansion, yielding to a correction in the form:

$$\tilde{E}_k = 4\pi \ln k + I + \frac{1}{2}W(l) + J_1(lk)^{-2} + o((lk)^{-2}). \quad (1.9)$$

By (1.7) and (1.9) a c.p.  $l(k)$  of  $\tilde{E}_k$  always exists provided  $J_1 > 0$ . First numerically, and then rigorously, we were disappointed to find that  $J_1 = 0$ .

Later on, we realized that  $-J_1$  is exactly the correlation coefficient  $A_0$  in (1.4) (with  $N = 3$  and  $m = 1$ ) introduced by Ovchinnikov and Sigal [13]. If  $u$  is a solution of (1.6) with vortices  $a_0 = 0$  and  $la_j, a_j = e^{\frac{2\pi i}{3}(j-1)}$  for  $j = 1, 2, 3$ , with  $n_0 = -1$  and  $n_1 = n_2 = n_3 = 1$ , then the function  $U(x) = u(\frac{x}{k})$  does solve

$$\begin{cases} -\Delta U = U(1 - |U|^2) & \text{in } B_k \\ U = g_0 & \text{on } \partial B_k \end{cases} \quad (1.10)$$

with vortices  $a_0$  and  $lka_j$  of vorticities  $n_0 = -1, n_1 = n_2 = n_3 = 1$ . Since (1.1) and (1.10) formally coincide when  $k = +\infty$ , it is natural to find a correlation term in the energy expansion

$\tilde{E}_k$  in the form  $-\frac{A_0}{a^2} = J_1(lk)^{-2}$ , where  $a = lk$  is the modulus of the  $lka_j$ 's for  $j = 1, 2, 3$ . Even more and not surprisingly, the function  $\tilde{U}_k(\frac{x}{k})$ , where

$$U_k[\eta(1 + \psi_0(k)) + (1 - \eta)e^{\psi_0(k)}]$$

is a very good approximating solution for (1.6) which improves the approximation rate of  $U_k$ , does coincide with the refined test functions used by Ovchinnikov and Sigal [13] to get the improved upper bound (1.3).

In conclusion, the vanishing of the correlation coefficient  $A_0$  does not support any conjecture concerning symmetry-breaking phenomena for (1.1) or the existence of collapsing vortices for (1.6) when  $k \rightarrow +\infty$ . Higher-order expansions would be needed in their study.

## 2. THE CORRELATION COEFFICIENT

Let  $N \geq 2$ . Let  $a_j = e^{\frac{2\pi i(j-1)}{N}}$ ,  $j = 1, \dots, N$ , be the  $N$ -roots of unity, and set  $n_j = m \in \mathbb{Z}$  for all  $j = 1, \dots, N$ ,  $a_0 = 0$  and  $n_0 = -\frac{N-1}{2}m$ . We aim to compute the correlation coefficient  $A_0 = A_0(m)$  given in (1.4). Since (in complex notation)  $\nabla\theta(x) = |x|^{-2}(-x_2, x_1)$  has the same modulus as  $\frac{1}{x} = \frac{\bar{x}}{|x|^2}$ , the correlation coefficient takes the form

$$A_0 = \frac{1}{4} \int_{\mathbb{R}^2} \left[ \left| \sum_{j=0}^N \frac{n_j}{x - a_j} \right|^4 - \sum_{j=0}^N \left| \frac{n_j}{x - a_j} \right|^4 \right]. \quad (2.1)$$

Since the integer  $m$  comes out as  $m^4$  from the expression (2.1), we have that  $A_0(m) = m^4 A_0(1)$ . Hereafter, we will assume  $m = 1$  and simply denote  $A_0(1)$  as  $A_0$ .

Let us first notice that  $A_0$  is not well-defined without further specifications, because the integral function in (2.1) is not integrable near the points  $a_j$ ,  $j = 0, \dots, N$ . Recall that the  $N$ -roots of unity  $a_1, \dots, a_N$  do satisfy the following symmetry properties:

$$\sum_{j=1}^N a_j^l = 0 \quad \forall |l| \leq N, l \neq 0, \quad (2.2)$$

as it can be easily deduced by the relation  $x^N - 1 = \prod_{j=1}^N (x - a_j)$ . A first application of (2.2) is the validity of

$$\sum_{j=1}^N \frac{1}{x - a_j} = \sum_{j=1}^N \frac{x^{N-1} + a_j x^{N-2} + \dots + a_j^{N-1}}{x^N - 1} = \frac{N x^{N-1}}{x^N - 1}, \quad (2.3)$$

which implies that the integral function in (2.1) near 0 has the form

$$\left| \sum_{j=0}^N \frac{n_j}{x - a_j} \right|^4 - \sum_{j=0}^N \left| \frac{n_j}{x - a_j} \right|^4 = -\frac{N(N-1)^3}{2} \operatorname{Re} \left( \frac{x^N}{(x^N - 1)|x|^4} \right) + O(1) \quad (2.4)$$

and is not integrable at 0 when  $N = 2$ . Similarly, setting  $\alpha_k(x) = -\frac{N-1}{2x} + \sum_{\substack{j=1 \\ j \neq k}}^N \frac{1}{x-a_j}$  for  $k = 1, \dots, N$ , near  $a_k$  we have that

$$\begin{aligned} \left| \sum_{j=0}^N \frac{n_j}{x-a_j} \right|^4 - \sum_{j=0}^N \left| \frac{n_j}{x-a_j} \right|^4 &= \frac{4}{|x-a_k|^4} \operatorname{Re}[(x-a_k)\alpha_k(x)] + \frac{2}{|x-a_k|^2} |\alpha_k(x)|^2 \quad (2.5) \\ &+ \left( 2\operatorname{Re} \frac{(x-a_k)\alpha_k(x)}{|x-a_k|^2} + |\alpha_k(x)|^2 \right)^2 - \frac{(N-1)^4}{16|x|^4} - \sum_{\substack{j=1 \\ j \neq k}}^N \frac{1}{|x-a_j|^4}. \end{aligned}$$

The function  $\alpha_k$  can not be computed explicitly, but we know that

$$\begin{aligned} \alpha_k(a_k) &= -\frac{N-1}{2a_k} + \sum_{\substack{j=1 \\ j \neq k}}^N \frac{1}{a_k - a_j} = a_k^{N-1} \left( -\frac{N-1}{2} + \sum_{j=2}^N \frac{1}{1-a_j} \right) \quad (2.6) \\ &= a_k^{N-1} \left( -\frac{N-1}{2} + \sum_{j=2}^N \frac{1 - \cos \frac{2\pi(j-1)}{N} + i \sin \frac{2\pi(j-1)}{N}}{2(1 - \cos \frac{2\pi(j-1)}{N})} \right) \\ &= i a_k^{N-1} \sum_{j=2}^N \frac{\sin \frac{2\pi(j-1)}{N}}{2(1 - \cos \frac{2\pi(j-1)}{N})} = 0 \end{aligned}$$

in view of  $\{a_j a_k^{N-1} : j = 1, \dots, N, j \neq k\} = \{a_2, \dots, a_N\}$  and the symmetry of  $\{a_1, \dots, a_N\}$  under reflections w.r.t. the real axis. By inserting (2.6) into (2.5) we deduce that the integral function in (2.1) near  $a_k$  has the form

$$\left| \sum_{j=0}^N \frac{n_j}{x-a_j} \right|^4 - \sum_{j=0}^N \left| \frac{n_j}{x-a_j} \right|^4 = \frac{4}{|x-a_k|^4} \operatorname{Re}[\alpha'_k(a_k)(x-a_k)^2] + O\left(\frac{1}{|x-a_k|}\right) \quad (2.7)$$

and is not integrable at  $a_k$  when  $\alpha'_k(a_k) \neq 0$ . Since the (possible) singular term in (2.4), (2.7) has vanishing integrals on circles, the meaning of  $A_0$  is in terms of a principal value:

$$A_0 = \frac{1}{4} \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^2 \setminus \cup_{k=0}^N B_\epsilon(a_k)} \left[ \left| \sum_{j=0}^N \frac{n_j}{x-a_j} \right|^4 - \sum_{j=0}^N \left| \frac{n_j}{x-a_j} \right|^4 \right]. \quad (2.8)$$

We would like to compute  $A_0$  in polar coordinates, even though the set  $\mathbb{R}^2 \setminus \cup_{k=0}^N B_\epsilon(a_k)$  is not radially symmetric. The key idea is to make the integral function in (2.8) integrable near any  $a_j$ ,  $j = 1, \dots, N$ , by adding suitable singular terms, in such a way that the integral in (2.8) will have to be computed just on the radially symmetric set  $\mathbb{R}^2 \setminus B_\epsilon(a_0)$ . To this aim, it is crucial to compute  $\alpha'_k(a_k)$ . Arguing as before, we get that

$$\begin{aligned} \alpha'_k(a_k) &= \frac{N-1}{2a_k^2} - \sum_{\substack{j=1 \\ j \neq k}}^N \frac{1}{(a_k - a_j)^2} = a_k^{N-2} \left( \frac{N-1}{2} - \sum_{j=2}^N \frac{1}{(1-a_j)^2} \right) \\ &= a_k^{N-2} \left( \frac{N-1}{2} - \sum_{j=2}^N \frac{(1 - \cos \frac{2\pi(j-1)}{N})^2 - \sin^2 \frac{2\pi(j-1)}{N}}{4(1 - \cos \frac{2\pi(j-1)}{N})^2} \right) \\ &= a_k^{N-2} \sum_{j=2}^N \frac{1}{2(1 - \cos \frac{2\pi(j-1)}{N})} = a_k^{N-2} \sum_{j=2}^N \frac{1}{|1-a_j|^2}. \quad (2.9) \end{aligned}$$

Since there holds  $\sum_{j=1}^{N-1} a_k^j = \sum_{j=2}^N a_j = -1$  for all  $k = 2, \dots, N$  in view of (2.2), we have that

$$\prod_{j=2}^N (z - a_j) = \frac{z^N - 1}{z - 1} = \sum_{p=0}^{N-1} z^p, \quad \prod_{\substack{j=2 \\ j \neq k}}^N (z - a_j) = \frac{\sum_{p=0}^{N-1} z^p}{z - a_k} = \sum_{p=0}^{N-2} z^p \sum_{l=0}^{N-2-p} a_k^l,$$

and then

$$\prod_{j=2}^N (1 - a_j) = N, \quad \prod_{\substack{j=2 \\ j \neq k}}^N (1 - a_j) = \sum_{l=0}^{N-2} (N - l - 1) a_k^l. \quad (2.10)$$

By (2.10) we get that

$$\begin{aligned} \beta_N &:= \sum_{j=2}^N \frac{4}{|1 - a_j|^2} = \sum_{j=2}^N \frac{4}{N^2} \prod_{\substack{k=2 \\ k \neq j}}^N |1 - a_k|^2 = \sum_{j=2}^N \frac{4}{N^2} \sum_{l,p=0}^{N-2} (N - l - 1)(N - p - 1) a_j^{l-p} \\ &= 4 \frac{N-1}{N^2} \sum_{l=1}^{N-1} l^2 - \frac{4}{N^2} \sum_{\substack{l,p=1 \\ l \neq p}}^{N-1} lp = \frac{4}{N} \sum_{l=1}^{N-1} l^2 - \frac{4}{N^2} \left( \sum_{l=1}^{N-1} l \right)^2 = \frac{2(N-1)(2N-1)}{3} - (N-1)^2 \\ &= \frac{N^2 - 1}{3} \end{aligned}$$

in view of (2.2). Since by (2.9)  $\alpha'_k(a_k) = \frac{\beta_N}{4} a_k^{N-2}$ , by (2.7) we have that

$$\left| \sum_{j=0}^N \frac{n_j}{x - a_j} \right|^4 - \sum_{j=0}^N \left| \frac{n_j}{x - a_j} \right|^4 - \sum_{j=1}^N \operatorname{Re} \left[ \frac{\beta_N a_j^2}{(x - a_j)^2 (1 + |x - a_j|^2)} \right] \in L^1(\mathbb{R}^2 \setminus \{0\}).$$

Since

$$\lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^2 \setminus \bigcup_{k=0}^N B_\epsilon(a_k)} \frac{a_j^2}{(x - a_j)^2 (1 + |x - a_j|^2)} = \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^2 \setminus B_\epsilon(a_j)} \frac{a_j^2}{(x - a_j)^2 (1 + |x - a_j|^2)} = 0,$$

we can re-write  $A_0$  as

$$\begin{aligned} A_0 &= \frac{1}{4} \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^2 \setminus B_\epsilon(0)} \left[ \left| \frac{(N+1)x^N + (N-1)}{2x(x^N - 1)} \right|^4 - \frac{(N-1)^4}{16|x|^4} - \sum_{j=1}^N \frac{1}{|x - a_j|^4} \right. \\ &\quad \left. - \sum_{j=1}^N \operatorname{Re} \left[ \frac{\beta_N a_j^2}{(x - a_j)^2 (1 + |x - a_j|^2)} \right] \right] \\ &= \frac{1}{4} \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^2 \setminus (B_\epsilon(0) \cup \{1-\epsilon \leq |x| \leq \frac{1}{1-\epsilon}\})} \left[ \left| \frac{(N+1)x^N + (N-1)}{2x(x^N - 1)} \right|^4 - \frac{(N-1)^4}{16|x|^4} - \sum_{j=1}^N \frac{1}{|x - a_j|^4} \right] \\ &\quad - \frac{1}{4} \operatorname{Re} \left[ \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^2 \setminus (B_\epsilon(0) \cup \{1-\epsilon \leq |x| \leq \frac{1}{1-\epsilon}\})} \sum_{j=1}^N \frac{\beta_N a_j^2}{(x - a_j)^2 (1 + |x - a_j|^2)} \right] =: \frac{1}{4} \mathbf{I} - \frac{1}{4} \mathbf{II} \quad (2.11) \end{aligned}$$

in view of (2.3).

As far as I, let us write the following Taylor expansions: for  $|x| < 1$  there hold

$$\begin{aligned} \frac{((N+1)x^N + (N-1))^2}{(1-x^N)^2} &= ((N-1)^2 + 2(N^2-1)x^N + (N+1)^2x^{2N}) \sum_{k \geq 0} (k+1)x^{kN} \\ &= (N-1)^2 + \sum_{k \geq 1} 4N(kN-1)x^{kN} = \sum_{k \geq 0} c_k x^{kN} \end{aligned} \quad (2.12)$$

and

$$\begin{aligned} \frac{((N-1)x^N + (N+1))^2}{(1-x^N)^2} &= ((N+1)^2 + 2(N^2-1)x^N + (N-1)^2x^{2N}) \sum_{k \geq 0} (k+1)x^{kN} \\ &= (N+1)^2 + \sum_{k \geq 1} 4N(kN+1)x^{kN} = \sum_{k \geq 0} d_k x^{kN}, \end{aligned} \quad (2.13)$$

where  $c_k = \max\{4N(kN-1), (N-1)^2\}$  and  $d_k = \max\{4N(kN+1), (N+1)^2\}$ . Letting  $\epsilon > 0$  small, by (2.12)-(2.13) we have that in polar coordinates (w.r.t. to the origin) I writes as

$$\begin{aligned} \text{I} &= \int_{\epsilon}^{1-\epsilon} \rho d\rho \int_0^{2\pi} d\theta \left[ \frac{1}{16\rho^4} \left| \sum_{k \geq 0} c_k \rho^{kN} e^{ikN\theta} \right|^2 - \frac{(N-1)^4}{16\rho^4} - \sum_{j=1}^N \left| \sum_{k \geq 0} (k+1) a_j^{k(N-1)} \rho^k e^{ik\theta} \right|^2 \right] \\ &\quad + \int_{\frac{1}{1-\epsilon}}^{\infty} \rho d\rho \int_0^{2\pi} d\theta \left[ \frac{1}{16\rho^4} \left| \sum_{k \geq 0} d_k \rho^{-kN} e^{-ikN\theta} \right|^2 - \frac{(N-1)^4}{16\rho^4} - \frac{1}{\rho^4} \sum_{j=1}^N \left| \sum_{k \geq 0} (k+1) a_j^k \rho^{-k} e^{-ik\theta} \right|^2 \right] \\ &\quad + o_{\epsilon}(1) \end{aligned}$$

with  $o_{\epsilon}(1) \rightarrow 0$  as  $\epsilon \rightarrow 0$ , in view of

$$|x - a_j|^{-4} = |a_j^{N-1}x - 1|^{-4} = \left| \sum_{k \geq 0} (k+1) a_j^{k(N-1)} x^k \right|^2, \quad |1 - a_j x|^{-4} = \left| \sum_{k \geq 0} (k+1) a_j^k x^k \right|^2$$

for  $|x| < 1$ . By the Parseval's Theorem we get that

$$\begin{aligned} \text{I} &= 2\pi \int_{\epsilon}^{1-\epsilon} \left[ \frac{1}{16} \sum_{k \geq 1} |c_k|^2 \rho^{2kN-3} - N \sum_{k \geq 0} (k+1)^2 \rho^{2k+1} \right] d\rho \\ &\quad + 2\pi \int_{\frac{1}{1-\epsilon}}^{\infty} \left[ \frac{1}{16} \sum_{k \geq 1} |d_k|^2 \rho^{-2kN-3} + \frac{(N+1)^4 - (N-1)^4}{16\rho^3} - N \sum_{k \geq 0} (k+1)^2 \rho^{-2k-3} \right] d\rho + o_{\epsilon}(1) \\ &= 2\pi N \int_0^{1-\epsilon} \left[ N \sum_{k \geq 0} (kN + N - 1)^2 \rho^{2kN+2N-3} - \sum_{k \geq 0} (k+1)^2 \rho^{2k+1} \right] d\rho \\ &\quad + 2\pi N \int_{\frac{1}{1-\epsilon}}^{\infty} \left[ N \sum_{k \geq 0} (kN + N + 1)^2 \rho^{-2kN-2N-3} - \sum_{k \geq 0} (k+1)^2 \rho^{-2k-3} \right] d\rho + N(N^2+1) \frac{\pi}{2} \\ &\quad + o_{\epsilon}(1) = 2\pi N \int_0^{1-\epsilon} \left[ N \sum_{k \geq 0} (kN + N - 1)^2 \rho^{2kN+2N-3} + N \sum_{k \geq 0} (kN + N + 1)^2 \rho^{2kN+2N+1} \right. \\ &\quad \left. - 2 \sum_{k \geq 0} (k+1)^2 \rho^{2k+1} \right] d\rho + N(N^2+1) \frac{\pi}{2} + o_{\epsilon}(1) \end{aligned}$$



as  $\epsilon \rightarrow 0$ . We compute now the integrals and let  $\epsilon \rightarrow 0$  to end up with

$$\begin{aligned} I &= 2\pi N \left[ \frac{N}{2} \sum_{k \geq 0} (kN + N - 1) \rho^{2kN+2N-2} + \frac{N}{2} \sum_{k \geq 0} (kN + N + 1) \rho^{2kN+2N+2} - \sum_{k \geq 0} (k+1) \rho^{2k+2} \right] \Big|_0^1 \\ &\quad + N(N^2 + 1) \frac{\pi}{2}. \end{aligned}$$

Denoting the function inside brackets as  $f(\rho)$ , we need now to determine the explicit expression of  $f(\rho)$  for  $\rho < 1$ :

$$\begin{aligned} f(\rho) &= \frac{N^2}{2} \rho^{2N-2} (1 + \rho^4) \sum_{k \geq 0} (k+1) (\rho^{2N})^k - \frac{N}{2} \rho^{2N-2} (1 - \rho^4) \sum_{k \geq 0} (\rho^{2N})^k - \rho^2 \sum_{k \geq 0} (k+1) (\rho^2)^k \\ &= \frac{N^2}{2} \rho^{2N-2} \frac{1 + \rho^4}{(1 - \rho^{2N})^2} - \frac{N}{2} \rho^{2N-2} \frac{1 - \rho^4}{1 - \rho^{2N}} - \frac{\rho^2}{(1 - \rho^2)^2} \\ &= \frac{1}{2} \frac{N^2 \rho^{2N-2} (1 + \rho^4) - N \rho^{2N-2} (1 - \rho^4) (1 - \rho^{2N}) - 2 \rho^2 \left( \sum_{j=0}^{N-1} \rho^{2j} \right)^2}{(1 - \rho^{2N})^2}, \end{aligned}$$

and then by the l'Hôpital's rule we get that

$$\begin{aligned} 4N^2 f(1) &= 2 \lim_{\rho \rightarrow 1} \frac{N(N-1) \rho^{N-1} + N(N+1) \rho^{N+1} - 2 \rho \left( \sum_{j=0}^{N-1} \rho^j \right)^2 + N \rho^{2N-1} - N \rho^{2N+1}}{(1 - \rho)^2} \\ &= \lim_{\rho \rightarrow 1} \frac{-N^2(N-2) \rho^{N-2} - N^2(N+2) \rho^N + 2 \left( \sum_{j=0}^{N-1} \rho^j \right)^2 + 4 \rho \left( \sum_{j=0}^{N-1} \rho^j \right) \left( \sum_{j=0}^{N-2} (j+1) \rho^j \right)}{1 - \rho} \\ &\quad + N \lim_{\rho \rightarrow 1} \frac{(2N+1) \rho^{2N} - (2N-1) \rho^{2N-2} - \rho^{N-2} - \rho^N}{1 - \rho} = -\frac{N^2(N^2 + 5)}{3}. \end{aligned}$$

In conclusion, for I we get the value

$$I = \frac{\pi}{3} N(N^2 - 1). \quad (2.14)$$

**Remark 2.1.** In [13] the value of  $A_0$  was computed neglecting the term II in (2.11). By (2.14) notice that  $\frac{m^4}{4} I = \frac{\pi}{12} m^4 N(N^2 - 1)$  does coincide with  $8\pi$  when  $(N, m) = (2, 2)$  and  $80\pi$  when  $(N, m) = (4, 2)$ , in agreement with the computations in [13].

As far as II, let us compute in polar coordinates the value of

$$\lim_{\epsilon \rightarrow 0} \sum_{j=1}^N \int_{\mathbb{R}^2 \setminus (B_\epsilon(0) \cup \{1-\epsilon \leq |x| \leq \frac{1}{1-\epsilon}\})} \frac{a_j^2}{(x - a_j)^2 (1 + |x - a_j|^2)} = \lim_{\epsilon \rightarrow 0} \int_{(0, 1-\epsilon) \cup (\frac{1}{1-\epsilon}, +\infty)} \rho \Gamma(\rho) d\rho,$$

where the function  $\Gamma$  is defined in the following way:

$$\begin{aligned} \Gamma(\rho) &= \sum_{j=1}^N \int_0^{2\pi} \frac{a_j^2}{(\rho e^{i\theta} - a_j)^2 (2 + \rho^2 - a_j \rho e^{-i\theta} - a_j^{N-1} \rho e^{i\theta})} d\theta \\ &= \frac{i}{\rho} \sum_{j=1}^N a_j^3 \int_\gamma \frac{dw}{(\rho w - a_j)^2 (w^2 - \frac{2+\rho^2}{\rho} a_j w + a_j^2)}, \end{aligned}$$

with  $\gamma$  the counterclockwise unit circle around the origin. Since

$$w^2 - \frac{2+\rho^2}{\rho}a_j w + a_j^2 = \left(w - \frac{2+\rho^2}{2\rho}a_j\right)^2 + a_j^2 \left(1 - \left(\frac{2+\rho^2}{2\rho}\right)^2\right),$$

observe that  $w^2 - \frac{2+\rho^2}{\rho}a_j w + a_j^2$  vanishes at  $\rho_{\pm}a_j$ , with

$$\rho_{\pm} = \frac{2+\rho^2}{2\rho} \pm \sqrt{\left(\frac{2+\rho^2}{2\rho}\right)^2 - 1}$$

satisfying  $\rho_- < 1 < \rho_+$  in view of  $\frac{2+\rho^2}{2\rho} \geq \sqrt{2}$ . Since

$$\left(\frac{1}{w^2 - \frac{2+\rho^2}{\rho}a_j w + a_j^2}\right)' \left(\frac{a_j}{\rho}\right) = a_j^{N-3} \rho^5,$$

by the Cauchy's residue Theorem the function  $\Gamma(\rho)$  can now be computed explicitly as

$$\begin{aligned} \Gamma(\rho) &= \frac{i}{\rho^3} \sum_{j=1}^N a_j^3 \int_{\gamma} \frac{dw}{(w - \frac{a_j}{\rho})^2 (w - \rho_- a_j)(w - \rho_+ a_j)} \\ &= 2\pi N \begin{cases} (\rho\rho_- - 1)^{-2}(\rho\rho_+ - \rho\rho_-)^{-1} & \text{if } \rho < 1 \\ (\rho\rho_- - 1)^{-2}(\rho\rho_+ - \rho\rho_-)^{-1} - \rho^2 & \text{if } \rho > 1. \end{cases} \end{aligned}$$

Since we have that

$$(\rho\rho_- - 1)^2 = \frac{1}{4}(\rho^2 - \sqrt{\rho^4 + 4})^2 = \frac{1}{2}(\rho^4 + 2 - \rho^2\sqrt{\rho^4 + 4}), \quad \rho\rho_+ - \rho\rho_- = \sqrt{\rho^4 + 4},$$

we get that

$$(\rho\rho_- - 1)^{-2}(\rho\rho_+ - \rho\rho_-)^{-1} = \frac{2}{(\rho^4 + 2)\sqrt{\rho^4 + 4} - \rho^2(\rho^4 + 4)} = \frac{\rho^4 + 2}{2\sqrt{\rho^4 + 4}} + \frac{\rho^2}{2},$$

and the expression of  $\Gamma(\rho)$  now follows in the form

$$\Gamma(\rho) = \pi N \frac{\rho^4 + 2}{\sqrt{\rho^4 + 4}} - \pi N \rho^2 + \begin{cases} 2\pi N \rho^2 & \text{if } \rho < 1 \\ 0 & \text{if } \rho > 1. \end{cases} \quad (2.15)$$

Note that

$$\rho \left( \frac{\rho^4 + 2}{\sqrt{\rho^4 + 4}} - \rho^2 \right) = \frac{4\rho}{(\rho^4 + 2)\sqrt{\rho^4 + 4} + \rho^2(\rho^4 + 4)}$$

is integrable in  $(0, \infty)$ , and we have that

$$\begin{aligned} \int_0^\infty \rho \left( \frac{\rho^4 + 2}{\sqrt{\rho^4 + 4}} - \rho^2 \right) d\rho &= \lim_{M \rightarrow +\infty} \frac{1}{2} \int_0^M \left( \frac{s^2 + 2}{\sqrt{s^2 + 4}} - s \right) ds \\ &= \lim_{M \rightarrow +\infty} \left[ \frac{s}{4} \sqrt{s^2 + 4} \Big|_0^M - \frac{M^2}{4} \right] = \lim_{M \rightarrow +\infty} \frac{M}{4} (\sqrt{M^2 + 4} - M) = \frac{1}{2}. \end{aligned} \quad (2.16)$$

Thanks to (2.15)-(2.16) we can compute

$$\lim_{\epsilon \rightarrow 0} \int_{(0, 1-\epsilon) \cup (\frac{1}{1-\epsilon}, +\infty)} \rho \Gamma(\rho) d\rho = \int_0^{+\infty} \rho \Gamma(\rho) d\rho = \pi N,$$

and for II we get the value

$$\text{II} = \frac{\pi}{3} N(N^2 - 1). \quad (2.17)$$

Finally, inserting (2.14) and (2.17) into (2.11) we get that the correlation coefficient vanishes:  $A_0 = 0$ . Then, there holds  $A_0(m) = 0$  for all  $m \in \mathbb{Z}$ , as claimed.

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